

General Theory of Overmeasurement of Discrete Quantum Observables and Application to Simultaneous Measurement

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Abstract. A complete theory of overmeasurement by measuring refinements of observables is presented. It encompasses a wider set of functions of observables (coarsenings) . Thus the theory has a broad potential application. It is applied to a thorough investigation of simultaneous measurements. In particular, the set of all simultaneous measurements for a given pair of compatible observables is determined.

Keywords Measurement. Functions of observables. Compatible observables.

1 Introduction

It is a textbook claim that two compatible discrete observables, i. e., ones of which the Hermitian operators representing them commute and have no continuous parts in their spectra, can be simultaneously measured. And this is done, so it is further claimed, by finding a common eigenbasis of the two operators and by measuring in which of the basis states the system is. The eigenvalues of the two operators that correspond to the measured basis state are then the simultaneous results of the measurement. This is a superficial and incomplete but typical presentation of simultaneous measurement.

The two discrete observables are *overmeasured* by a common overmeasurement, though this term is usually not used. In this article a *complete theory of overmeasurement* is expounded together with a complete theory of simultaneous measurements as an application.

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If one defines general exact measurement, following [1], by the *calibration condition* (see relation (4) below), then overmeasurement is the *most general exact measurement*. The opposite of overmeasurement is undermeasurement. Since the points of a continuous spectrum cannot be measured, they must be undermeasured. Von Neumann in his famous book [2] (cf chapter III, section 3. p. 220 there) explains this, though he does not use the term "undermeasurement". (His term for undermeasurement is "measurement with only limited accuracy".)

It is hoped that the complete theory of overmeasurement that is to be presented will not only give a deeper conceptual insight in measurement theory, but also find new applications.

The investigation is restricted to *discrete observables* in this article. They will always be given in their *unique spectral form* (unless otherwise stated), which means, by definition, that there is no repetition in the eigenvalues $\{o_k : \forall k\}$ that are displayed in the spectral form:

$$O_A = \sum_k o_k E_A^k, \quad (1a)$$

so that $\{E_A^k : \forall k\}$ are the corresponding eigen-projectors. The index A denotes the measured subsystem. The spectral form is accompanied by the spectral (orthogonal projector) decomposition of the identity operator I_A (also called the "completeness relation")

$$\sum_k E_A^k = I_A. \quad (1b)$$

When O_A is measured in a suitable interaction with a measuring instrument B, then an initial or ready-to-measure state $|\phi\rangle_B^i$ together with a so-called *pointer observable*

$$P_B = \sum_k p_k F_B^k \quad (2)$$

are given. The eigen-projectors $\{F_B^k : \forall k\}$ are metaphorically called —pointer positions—. Also they satisfy the completeness relation $\sum_k F_B^k = I_B$. Notice the co-indexing in (2) and (1a) based on a one-to-one relation between the possible measurement results $\{o_k : \forall k\}$ and all possible pointer positions.

The suitable measurement interaction is assumed to be incorporated in a unitary operator U_{AB} , which maps the composite initial state to the final state $|\Phi\rangle_{AB}^f$

$$|\Phi\rangle_{AB}^f = U_{AB} \left(|\phi\rangle_A^i |\phi\rangle_B^i \right), \quad (3)$$

where $|\phi\rangle_A^i$ is an arbitrary initial state of the object subsystem.

This is the basic formalism of unitary measurement theory, or premeasurement theory or measurement theory short of collapse [1], [3], [4]. The general unitary (also called "exact") measurements of discrete observables are defined by the *calibration condition*, which requires that if the object has a sharp value $o_{\bar{k}}$ of the measured observable in the initial state, then the final composite state has the corresponding sharp pointer position $F_B^{\bar{k}}$:

$$E_A^{\bar{k}} |\phi\rangle_A^i = |\phi\rangle_A^i \Rightarrow F_B^{\bar{k}} |\Phi\rangle_{AB}^f = |\Phi\rangle_{AB}^f. \quad (4)$$

(Note that the mutually equivalent eigenvalue equations $E_A^{\bar{k}} |\phi\rangle_A^i = |\phi\rangle_A^i$ and $O_A |\phi\rangle_A^i = o_{\bar{k}} |\phi\rangle_A^i$ are the standard way to express certainty in quantum mechanics, and " \Rightarrow " stands for logical implication.)

In this study we will not treat the important special case of nondemolition (synonyms: repeatable, predictive, first-kind) measurements, nor the much used even more special special case of ideal measurements [5].

It is known from von Neumann's book [2] that an observable O_A given by (1a) can be measured by measuring a complete observable, i. e., one with no degeneracy in any of its eigenvalues,

$$O_A^r = \sum_k \sum_{n_k} o_{k,n_k}^r |k, n_k\rangle \langle k, n_k|, \quad (5a)$$

$$(k, n_k) \neq (k', n_{k'}) \Rightarrow o_{k,n_k}^r \neq o_{k',n_{k'}}^r, \quad (5b)$$

which is a so-called *refinement* of O_A , i. e., for which

$$\forall k : \sum_{n_k} |k, n_k\rangle \langle k, n_k| = E_A^k \quad (5c)$$

is valid.

One is dealing with overmeasurement of O_A , where actually O_A^r is measured, and, if. e. g., o_{k,n_k}^r is the result of measurement, then by quantum -logical implication, due to $|k, n_k\rangle \langle k, n_k| \leq E_A^k$ (symbolic for $|k, n_k\rangle \langle k, n_k| E_A^k = |k, n_k\rangle \langle k, n_k|$), also the pointer position E_A^k has occurred or the result o_k of O_A ($= \sum_{k'} o_{k'} E_A^{k'}$) is obtained.

2 General Theory of Overmeasurement

The unitary quantum formalism is restricted to unitary evolutions, and, as well known, it cannot in general derive the (unknown) final state $(|\Phi\rangle_{AB}^f)^k$ of *complete measurement*, which includes collapse to the definite result p_k or, equivalently, the occurrence of the pointer position F_B^k . But the very fact that it contains the information of a definite o_k result, i. e., due to $(\langle \Phi|_{AB}^f)^k F_B^k (|\Phi\rangle_{AB}^f)^k = 1$, one must have equivalently,

$$(|\Phi\rangle_{AB}^f)^k = F_B^k (|\Phi\rangle_{AB}^f)^k. \quad (6)$$

The final state (6) of complete measurement might even be mixed. For simplicity we restrict it to a pure state.

2.1 Overmeasurement - The formal part

Overmeasurement is usually defined in a more narrow sense by any single-valued function $f(\dots)$ on the real axis. It determines an observable \bar{O}_A that is the corresponding

function of the given observable O_A ($= \sum_k o_k E_A^k$):

$$\begin{aligned} \bar{O}_A &\equiv f(O_A) \equiv \sum_k f(o_k) E_A^k = \\ &\sum_l \bar{o}_l E_A^l, \quad l \neq l' \Rightarrow \bar{o}_l \neq \bar{o}_{l'}. \end{aligned} \quad (7a)$$

Note that the first spectral form in (7a), unlike the second one, is, in general, non-unique.

In the context of overmeasurement, \bar{O}_A is called a coarsening or a coarser observable, and O_A is said to be a refinement or a finer observable. (These terms are meant in the improper sense. For instance, "finer" is actually "properly finer" or equal.)

The indices l are defined so as to make the spectral form of the coarser observable \bar{O} unique. This implies that the index set $\{\forall k\}$ in the unique spectral form of the finer observable O is broken up into equivalence classes: $\{\forall k\} = \sum_l C_l$. In other words, it can be viewed as the union of non-intersecting subsets (classes) C_l . Belonging to the same class C_l is defined as follows.

$$\forall l: k, k' \in C_l \Leftrightarrow f(o_k) = f(o_{k'}) = \bar{o}_l. \quad (7b)$$

Thus $f(\dots)$, primarily given as a function on the real axis, determines a function, we denote it by the same symbol f , mapping the index set $\{\forall k\}$ onto the new index set $\{\forall l\}$. Note that the *inverse multivalued function* f^{-1} takes the latter index set onto the former and its images are precisely the mentioned equivalence classes:

$$\forall l: k, k' \in C_l \text{ if and only if } k, k' \in f^{-1}(l). \quad (7c)$$

It is sometimes useful to define *overmeasurement in a broader sense* by an (arbitrary) single-valued map f taking the index set $\{\forall k\}$ of the finer observable O onto the index set $\{\forall l\}$ of the coarser observable \bar{O} . But always the essential thing is the relation

$$\forall l: E_A^l = \sum_{k, f(k)=l} E_A^k. \quad (8a)$$

Relation (8a) follows from (7a) if one has the narrower definition, and it is the most important part of the definition of overmeasurement in the broader definition. In the latter case the eigenvalues $\{\bar{o}_l\}$ of the coarser observable need not be related to those of the finer observable.

As a consequence of (8a), the orthogonality of the projectors $\{E_A^k: \forall k\}$ leads to

$$E_A^l E_A^k = 0 \text{ if } f(k) \neq l. \quad (8b)$$

Parallely with the unique spectral form of the coarser observable, also the unique spectral form

$$\bar{P}_B = \sum_l \bar{p}_l F_B^l, \quad (9)$$

of the pointer observable of the coarsening is going to play an important role. Note that the eigenvalues $\{\bar{p}_l : \forall l\}$ can be arbitrary distinct real numbers. Further, by definition

$$\forall l : F_B^l = \sum_{k, f(k)=l} F_B^k, \quad (10a)$$

where the function $f : \{\forall k\} \rightarrow \{\forall l\}$ is the one that determines the coarsening $O \rightarrow \bar{O}$. Relations (10a) are symmetrical to (8a).

One has also

$$F_B^l F_B^k = F_B^k \quad \left(\Leftrightarrow \quad F_B^l \geq F_B^k \right) \quad \text{if} \quad f(k) = l. \quad (10b)$$

Naturally, the eigen-projectors of the finer and of the coarser observable and the eigen-projectors of the corresponding pointer observables satisfy symmetrical relations. But we have written down only those that we shall make use of.

2.2 Overmeasurement - The physical part

Now we make *the first physical step* showing that *any* unitary measurement of an observable O_A is by this very fact *a unitary measurement also of any coarser observable* \bar{O}_A related to the finer observable O_A by a given map of the index set of the latter onto that of the coarser observable. In particular, we shall demonstrate that the calibration condition, which is by definition valid for the measurement of the finer observable, implies that also the calibration condition for the coarser observable is satisfied.

We assume that the initial state $|\phi\rangle_A^i$ of the object has a sharp value \bar{o}_l of the coarser observable:

$$|\phi\rangle_A^i = E_A^{\bar{l}} |\phi\rangle_A^i. \quad (11)$$

Utilizing the completeness relation $I_A = \sum_k E_A^k$ in the decomposition $|\phi\rangle_A^i = \sum_k E_A^k |\phi\rangle_A^i$ and (11), $|\Phi\rangle_{AB}^f$, which is defined by (3), becomes equal to

$$\sum_k \|E_A^k |\phi\rangle_A^i\| U_{AB} \left[\left(E_A^k E_A^{\bar{l}} |\phi\rangle_A^i / \|E_A^k |\phi\rangle_A^i\| \right) |\phi\rangle_B^i \right]. \quad (12)$$

Since the sum can be broken up $\sum_k \dots = \sum_{k, f(k) \neq \bar{l}} \dots + \sum_{k, f(k) = \bar{l}} \dots$, (8b) makes the first sum zero. Hence, making use of the assumption that the measurement of the finer observable satisfies the calibration condition in the form of inserting F_B^k , and using (11) again to suppress $E_A^{\bar{l}}$, $|\Phi\rangle_{AB}^f$ is further equal to

$$\sum_{k, f(k) = \bar{l}} \|E_A^k |\phi\rangle_A^i\| F_B^k U_{AB} \left[\left(E_A^k |\phi\rangle_A^i / \|E_A^k |\phi\rangle_A^i\| \right) |\phi\rangle_B^i \right]. \quad (13)$$

Finally, taking into account (10b), we obtain

$$F_B^{\bar{l}} |\Phi\rangle_{AB}^f = |\Phi\rangle_{AB}^f, \quad (14)$$

, which expresses certainty. This proves the claim. Thus, in view of (11) and (14), the calibration condition is valid for the overmeasurement of the coarser observable.

Naturally, due to the usual convention, if $E_A^k |\phi\rangle_A^i = 0$ in some term, then the expression that follows in the same term need not be defined; the term is by definition zero.

Now we can make *the second physical step concerning the result of complete measurement*. The claim is that *if the complete measurement of the finer observable produces the result o_k , then this same process of measurement gives the result $\bar{o}_{f(k)}$ for the coarser observable*. The proof is an immediate consequence of (10b). Namely, putting $l \equiv f(k)$, one obtains

$$\begin{aligned} F_B^l \{|\Phi\rangle_{AB}^f\}^k &= F_B^l \left(F_A^k \{|\Phi\rangle_{AB}^f\}^k \right) = \left(F_B^l F_A^k \right) \{|\Phi\rangle_{AB}^f\}^k = \\ &F_A^k \{|\Phi\rangle_{AB}^f\}^k = \{|\Phi\rangle_{AB}^f\}^k. \end{aligned} \quad (15)$$

We have thus proved that the final state $\{|\Phi\rangle_{AB}^f\}^k$ of complete measurement has the definite result $\bar{o}_{l \equiv f(k)}$ of the coarser observable. If this final state is mixed, the proof is analogous, but it requires certain generalizations of the formalism. Hence it is omitted for simplicity.

If a coarser observable \bar{O}_A in the proper sense is given first, there exist various refinements; there can even be refinements of refinements. And one has transitivity: a refinement of a refinement is a refinement of the coarsest observable \bar{O}_A . Therefore one can speak of *degrees of overmeasurement* of the given observable \bar{O}_A .

The two *extreme degrees* are: *minimal measurement*, when there is actually no refinement, and *maximal overmeasurement*, when the measured finer observable O_A is a complete observable, i. e., one all eigenvalues of which are non-degenerate (cf the end of the Introduction).

The best known example of minimal measurement is ideal measurement, also called Lüders or von Neumann-Lüders measurement (cf section 7 in [3]).

Minimal measurement in a general sense was introduced by the present author [6]. Maximal overmeasurement is also called measurement in a given basis (having in mind the eigen-basis of the complete observable; its eigenvalues anyway play no role in measurement theory).

One should note that, if minimal measurement is included in overmeasurement (as the trivial, improper extreme), then every measurement is an overmeasurement.

3 Simultaneous measurement

This section is devoted to an illustration of application of overmeasurement to a topic that is well known but not well proved and not well understood in its fine details.

To begin with, let us *define* that by *simultaneous measurement* of two observables $O'_A \left(= \sum_m o_m E_A^m \right)$ and $O''_A \left(= \sum_n o_n E_A^n \right)$ is understood measurement of one observable $O_A \left(= \sum_k o_k E_A^k \right)$ that is so chosen that any result o_k implies (by quantum-logical implication) a result $o_{m(k)}$ of O'_A and simultaneously a result $o_{n(k)}$ of O''_A . Besides, each possible result o_m of O'_A and o_n of O''_A must be thus obtainable for some initial state $|\phi\rangle_A^i$.

3.1 Common overmeasurement and compatibility

The very definition of simultaneous measurement implies that, by *necessity*, there must exist two functions f' and f'' mapping the set of all indices $\{\forall k\}$ onto the sets of all indices $\{\forall m\}$ and $\{\forall n\}$ respectively so that using the notation

$$\forall k : \quad f'(k) = m(k), \quad f''(k) = n(k), \quad (16a)$$

one has

$$\begin{aligned} \forall k : \quad E_A^k &\leq E_A^{m(k)} \quad \left(E_A^k E_A^{m(k)} = E_A^k \right) \quad \text{and} \\ E_A^k &\leq E_A^{n(k)} \quad \left(E_A^k E_A^{n(k)} = E_A^k \right). \end{aligned} \quad (16b)$$

It is seen that O_A must be a *common refinement* of O'_A and O''_A and hence the measurement of O_A a common overmeasurement of the latter two observable. Thus, *necessity* of the common refinement claim is proved.

As to proving *sufficiency* of the stated claim, it clearly follows from the definition of simultaneous measurement that any common overmeasurement will achieve it. \square

Furthermore, relations (16a,b) imply

$$\begin{aligned} \forall m : \quad E_A^m &= \sum_{k \in (f')^{-1}(m)} E_A^k \\ \text{and } \forall n : \quad E_A^n &= \sum_{k \in (f'')^{-1}(n)} E_A^k, \end{aligned} \quad (17)$$

which, in turn, has

$$\forall m, n : \quad [E_A^m, E_A^n] = 0 \quad (18)$$

as its consequence.

Two observables that satisfy the commutativity condition (18) are said to be *compatible*. In this way it is proved that for simultaneous measurability compatibility is *necessary*. We now prove that it is also *sufficient*.

Assuming the validity of (18), each product $E_A^m E_A^n$ is a projector, and

$$(E_A^m E_A^n)(E_A^{m'} E_A^{n'}) = (E_A^m E_A^{m'})(E_A^n E_A^{n'}) =$$

$$\delta_{m,m'}\delta_{n,n'}E_A^mE_A^n,$$

i. e., any two projectors in the set $\{E_A^mE_A^n : \forall m, \forall n\}$ are orthogonal. Finally, multiplying the two completeness relations $\sum_m E_A^m = O_A$ and $\sum_n E_A^n = I_n$, one obtains the completeness relation $\sum_m \sum_n E_A^m E_A^n = I_A$.

Let us enumerate by k all non-zero distinct projectors

$$E_A^k \equiv E_A^m E_A^n \neq 0, \quad (19a)$$

and take an arbitrary set $\{o_k : \forall k\}$ of distinct real numbers. Then, it is obvious from the arguments above, that

$$O_A \equiv \sum_k o_k E_A^k \quad (19b)$$

is a common refinement of O'_A and O''_A . Hence its measurement is a common over-measurement of these two given observables. \square

If two observables O'_A and O''_A are bounded, then they are compatible if and only if they commute $[O'_A, O''_A] = 0$. Also this claim is, unlike its proof, well known. (For the reader's convenience we prove it in Appendix B.)

Incidentally, it is known in linear analysis, or rather from the theory of at most countably infinite complex Hilbert spaces [2], that if the spectrum $\{o_m : \forall m\}$ of a given observable O'_A ($= \sum_m o_m E_A^m$) is known, then a useful necessary and sufficient condition for boundedness of O'_A is that the spectrum belongs to a finite closed interval:

$$\{o_m : \forall m\} \subset [a, b], \quad a < b, \quad a, b \text{ real numbers.} \quad (20)$$

3.2 The set of all simultaneous measurements

In this subsection we prove the following *claim*. Let two compatible observables O_A ($= \sum_m o_m E_A^m$) and O'_A ($= \sum_n o_n E_A^n$) (cf definition in relation (18)) be given, and let us understand the concept of "refinement" in the improper sense (cf last passage in section 2). Then an observable \bar{O}_A ($= \sum_l o_l G_A^l$) is their *common refinement if and only if* it is a refinement of the observable O_A^M ($= \sum_k o_k E_A^k$) defined by relations (19a) and (19b). The latter observable is thus the *maximal common refinement* of the given two compatible observables O_A and O'_A .

Since any refinement of a refinement is a refinement, also any refinement of a common refinement is a common refinement. Thus *sufficiency* easily follows.

To prove *necessity*, we assume that an observable \bar{O}_A ($= \sum_l o_l G_A^l$) is a common refinement of the given two observables O_A and O'_A . This implies that there are two

surjections (onto maps)

$$\begin{aligned} \bar{f}' : \{\forall l\} &\rightarrow \{\forall m\}, \quad \bar{f}'' : \{\forall l\} \rightarrow \{\forall n\} \text{ such that} \\ \forall l : G_A^l &\leq E_A^{m \equiv \bar{f}'(l)}, \quad G_A^l \leq E_A^{n \equiv \bar{f}''(l)}. \end{aligned} \quad (21)$$

Note that $\forall l : E_A^{m \equiv \bar{f}'(l)} E_A^{n \equiv \bar{f}''(l)} \neq 0$ because G_A^l is a non-zero common lower bound of the two factors. Hence we can define an injection (into map) of the index set $\{\forall l\}$ into the index set $\{\forall k\}$:

$$\begin{aligned} f \equiv \bar{f}', \bar{f}'' : \forall l : k(l) \equiv f(l) &\equiv [m \equiv \bar{f}'(l)], [n \equiv \bar{f}''(l)] \\ \Rightarrow G_A^l &\leq E_A^{k(l)}. \end{aligned} \quad (22)$$

In section 2 we have seen that overmeasurement is based on measuring a refinement. Relation (22) would prove \bar{O}_A to be a refinement of O_A^M if it were a surjection of $\{\forall l\}$ onto $\{\forall k\}$.

In Appendix A it is shown that $\sum_l G_A^l \leq \sum_k E_A^k$ (cf relation (A.5)). Since the observable \bar{O}_A has its completeness relation $\sum_l G_A^l = I_A$, we have $I_A \leq \sum_k E_A^k$. Since I_A is an upper bound of all projectors, we have $I_A \leq \sum_k E_A^k \leq I_A$ implying $\sum_k E_A^k = I_A$. Hence, after all, we are dealing with a surjection and the necessity of the claim \bar{O}_A being a refinement of O_A^M is proved. \square

Incidentally, the products $E_A^m E_A^n$ outside the the image $\bar{f}(\{\forall l\})$ in $\{\forall m, n\}$ must be all zero on account of the orthogonality of the eigen-projectors.

Every complete observable O_A^C (cf (5a-c)) that is a refinement of the maximal common refinement O_A^M given by (19a) and (19b) for two given compatible observables is a *local minimum* in the set of all common refinements. By definition this means that O_A^C has no refinement. This is in contrast with O_A^M , which is a global maximum.

3.3 Corollaries

COROLLARY 1 Let $\{O_A^q = \sum_{n_q} o_{n_q} E_A^{n_q} : q = 1, 2, \dots, Q\}$ be an arbitrary set of Q (a natural number) pairwise compatible discrete observables in their unique spectral forms. The maximal common refinement O_A^M is defined in its unique spectral form as follows.

$$O_A^M \equiv \sum_{n_1} \sum_{n_2} \dots \sum_{n_Q} o_{n_1 \dots n_Q} \prod_{q \in Q} E_A^{n_q}, \quad (23)$$

where it is understood that all terms in which the projectors multiply into zero are omitted and all eigenvalues are arbitrary but distinct.

Simultaneous measurement of all observables from the set is performed if and only if O_A^M or any of its refinements is measured.

PROOF For $Q = 2$ the claim has been proved in the preceding two subsections. Let us assume that it is valid for R observables, where R is a natural number. Then we know, again from the preceding two subsections, that for $R + 1$ observables the claim of Corollary 1 is valid. Hence, by total induction we conclude that the claim is valid for any natural number Q . \square

COROLLARY 2 Let $O_A \left(= \sum_k o_k E_A^k \right)$ be any discrete observable given in its unique spectral form. Further, let

$$\{\forall k\} = \sum_l \mathcal{C}_l \quad (24a)$$

be any breaking up the index set into classes, i. e., writing it as the union of non-intersecting subsets \mathcal{C}_l . Then, defining

$$\forall l : E_A^l \equiv \sum_{k \in \mathcal{C}_l} E_A^k \quad (24b)$$

any observable

$$\bar{O}_A \equiv \sum_l o_l E_A^l \quad (24c)$$

with arbitrary but distinct eigenvalues is a coarsening of O_A and any measurement of O_A is, at the same time, also a measurement, or rather an overmeasurement of the latter coarsened observable.

No careful reader of section 2 will need proof of Corollary 2.

4 Summing Up

The investigation in this article began with von Neumann's treatment of the measurement of any discrete observable via a suitably chosen complete one (cf relations (5a-c)). It was pointed out that the latter observable is a refinement, and its measurement is overmeasurement of the initially given observable.

Then a general and detailed theory of overmeasurement was presented in the hope that it will find applications.

Next, the study turned to simultaneous measurement, as to an important application of the concept of a refinement of an observable and of overmeasurement as a procedure. It turned out that simultaneous measurement is the same thing as common overmeasurement. To illustrate the power of overmeasurement theory, some fine points of simultaneous measurement, especially finding the set of all simultaneous measurements for a given pair of observables, have been worked out.

Appendix A: Some helpful projector relations

We assume that it is known that the set of all projectors in an at most countably-infinite

dimensional complex Hilbert space (state space of a quantum system) is a partially ordered set with the quantum-logical implication $E \leq F \left(\equiv EF = E \right)$. Besides it is a complete lattice, i. e., each non-empty subset has both a greatest lower bound (glb) and a least upper bound (lub). We now prove algebraically a few (more or less well known) claims that we make use of in subsection 3B.

If the reader knows that there exists a natural isomorphism between the partially ordered set of all projectors and that of all subspaces of the state space, then he may find it easier to supply the proofs in terms of subspaces. (This isomorphism maps a projector E into its range $\mathcal{R}(E)$. The inverse of this isomorphism takes any subspace S into the projector that makes S its range.)

PROOF of the claim

$$EF = FE \Rightarrow EF = \text{glb}(E, F). \quad (\text{A.1})$$

Let G be any common lower bound of E and F : $GE = GF = G$. Then $G(EF) = GF = G$. Thus, G is a lower bound also of EF as claimed.

PROOF of the claim that if $\{E_l : l = 1, 2, \dots, L\}$, where L may even be the power of a countably infinite set, is a set of pairwise orthogonal projectors, then

$$S \equiv \sum_{l=1}^L E_l = \text{lub}\{E_l : \forall l\}. \quad (\text{A.2})$$

The projector S is a common upper bound of the projectors in the given set because

$$\forall l : E_l S = \sum_{l'=1}^L E_l E_{l'} = E_l.$$

Let F be any common upper bound for all projectors E_l . Then it is also an upper bound of S because

$$SF = \left(\sum_{l=1}^L E_l \right) F = \sum_{l=1}^L E_l F = S.$$

PROOF of the claim that if two projectors E, F are orthogonal, and a third projector G implies (quantum-logically) one of them, then also G is orthogonal to the other projector:

$$EF = 0, \quad G \leq E, \quad \Rightarrow \quad GF = 0. \quad (\text{A.3})$$

This is so because

$$GF = (GE)F = G(EF) = 0.$$

PROOF of the claim that if one has two finite or infinite sums of pairwise orthogonal projectors, $(\sum_l G_l, \sum_k E_k)$ such that a map f is given that takes the index set $\{l\}$ into the index set $\{k\}$ so that

$$\forall l : G_l \leq E_{k=f(l)}, \quad (A.4)$$

then the former sum is a lower bound of the latter

$$\sum_l G_l \leq \sum_k E_k. \quad (A.5)$$

To begin the proof, we single out the subset $\{\bar{k} = f(l) : \forall l\} \left(\subseteq \{\forall k\} \right)$ that is the image of $\{\forall l\}$ regarding the map f . Next we break up $\{\forall l\}$ into classes $\mathcal{C}_{\bar{k}} \equiv f^{-1}(\bar{k}) : \{\forall l\} = \sum_{\bar{k}} \mathcal{C}_{\bar{k}}$. Then, we claim that

$$\forall \bar{k} : \mathcal{C}_{\bar{k}} E_{\bar{k}'} = \delta_{\bar{k}, \bar{k}'} \mathcal{C}_{\bar{k}}. \quad (A.6)$$

To prove the step (A.6), one has $\mathcal{C}_{\bar{k}} E_{\bar{k}} = \sum_{l \in f^{-1}(\bar{k})} G_l E_{\bar{k}} = \mathcal{C}_{\bar{k}}$ due to (A.4). As to the claimed (logical) implication, $(\bar{k} \neq \bar{k}') \Rightarrow \mathcal{C}_{\bar{k}} E_{\bar{k}'} = 0$, it follows from (A.3) because $E_{\bar{k}} E_{\bar{k}'} = 0$, and $\mathcal{C}_{\bar{k}} \leq E_{\bar{k}}$.

Relation (A.6) implies

$$\sum_l G_l = \sum_{\bar{k}} \mathcal{C}_{\bar{k}} \leq \sum_{\bar{k}} E_{\bar{k}} \quad (A.7)$$

because $(\sum_l G_l)(\sum_{\bar{k}} E_{\bar{k}}) = \sum_{\bar{k}, \bar{k}'} \mathcal{C}_{\bar{k}} E_{\bar{k}} = \sum_{\bar{k}} \mathcal{C}_{\bar{k}} = \sum_l G_l$.

Next, the orthogonality of the projectors E_k implies

$$\sum_{\bar{k}} E_{\bar{k}} \leq \sum_k E_k = \sum_{\bar{k}} E_{\bar{k}} + \dots \quad (A.8)$$

Hence, the transitivity of quantum-logical implication supplies the final proof of (A.5) $\sum_l G_l \leq \sum_k E_k$.

Appendix B: On compatibility of bounded observables

The main claim of Appendix B is a consequence of the following *more general claim*:

CLAIM 1. Let $O = \sum_k o_k E_k$ be a bounded discrete observable in its unique spectral form and let \bar{O} be a bounded linear operator. Then the following three relations are *equivalent*:

$$\begin{aligned} (1) \quad [O, \bar{O}] = 0 & \Leftrightarrow (2) \quad \bar{O} = \sum_k E_k \bar{O} E_k \\ & \Leftrightarrow (3) \quad \forall k : [E_k, \bar{O}] = 0. \end{aligned} \quad (B.1)$$

PROOF of the claimed (logical) implications in (B.1) will be given in an in-circle way as follows: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

(1) \Rightarrow (2) : One can write (1) in (B.1), on account of $I = \sum_k E_k$, as follows, and, multiplying out the factors in the commutator, one obtains

$$\begin{aligned} & \left[\sum_k o_k E_k, \sum_k \sum_{k'} E_k \bar{O} E_{k'} \right] = 0 \Rightarrow \\ & \sum_k \sum_{k'} o_k E_k \bar{O} E_{k'} - \sum_k \sum_{k'} o_{k'} E_k \bar{O} E_{k'} = 0 \\ & \Rightarrow \sum_{k \neq k'} (o_k - o_{k'}) E_k \bar{O} E_{k'} = 0. \end{aligned} \quad (B.2)$$

Taking fixed k and k' , we multiply the last relation by E_k from the left and by $E_{k'}$ from the right to obtain $(o_k - o_{k'}) E_k \bar{O} E_{k'} = 0$, and finally $\forall (k \neq k') : E_k \bar{O} E_{k'} = 0$. Thus, (2) follows from (1).

(2) \Rightarrow (3) : Multiplying $\bar{O} = \sum_k E_k \bar{O} E_k$ from the left or alternatively from the right by the same arbitrary fixed E_k , one obtains the same term $E_k \bar{O} E_k$. Hence (3) is a consequence of (2) in (B.1).

(3) \Rightarrow (1) : The third relation implies

$$[O, \bar{O}] = \sum_k o_k [E_k, \bar{O}] = 0. \quad (B.3)$$

This ends the proof.

Claim 1 implies the claim that we actually want to prove in this appendix.

CLAIM 2. Let $O = \sum_m o_m E_m$ and $O' = \sum_n o_n E_n$ be two discrete Hermitian operators given in their unique spectral forms. Then the two operators commute, $[O, O'] = 0$, if and only if each eigen-projector of the former commutes with each eigen-projector of the latter $\forall m, n : [E_m, E_n] = 0$.

PROOF. Sufficiency. Assuming $\forall m, n : [E_m, E_n] = 0$, one obtains $[O, O'] = 0$ as seen by substituting the unique spectral forms for both operators and utilizing the bilinearity of the commutator.

Necessity. According to the above proposition, (1) in (B.1) implies (3) in (B.1). Hence, $[O, O'] = 0$ implies $\forall m : [\hat{E}_m, O'] = 0$. A repeated application of the mentioned claim in the Proposition can now be written as

$$\forall m : [O', E_m] = 0 \Rightarrow \forall n : [E_n, E_m] = 0$$

□

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